IMO - Selection 2018

First exam - 12 May 2018

Duration: 4.5 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

1. Let $k \geq 0$ be an integer. Determine all polynomials P of degree k with real coefficients such that P has k distinct real roots and for any root a of P

$$P(a+1) = 1.$$

2. Let O be the circumcenter of an acute triangle ABC. The line OA intersects the altitude h_b at P and the altitude h_c at Q. Let H be the orthocenter of ABC. Prove that the circumcenter of PQH lies on the median of ABC through A.

Remark: the altitude h_a is the line through A perpendicular to BC.

3. Along the coast of a round island there are 20 different villages. Each of these villages has 20 fighters and all 400 fighters have different strengths.

Every pair of neighbouring villages A and B organises a competition during which all 20 fighters from village A battles individually against every fighter from the village B. The winner of a battle is always the stronger fighter. We say that the village A is *stronger* than the village B, if during at least k out of the 400 battles, a fighter from village A has won.

It turns out that every village is stronger than its clockwise neighbouring village. Determine the maximal value of k that allows such an outcome.

Good Luck!

IMO - Selection 2018

Second exam - 13 May 2018

Duration: 4.5 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

4. Let n be an even positive integer. We partition the numbers $1, 2, \ldots, n^2$ into two sets A and B with the same size such that all of the n^2 numbers belong to exactly one of the two sets. Let S_A and S_B be the sum of all the elements in A respectively B. Determine all n such that there is a partition with

$$\frac{S_A}{S_B} = \frac{39}{64}.$$

5. Let n be a positive integer. We consider an $n \times n$ grid. We colour k squares in black, such that given any three columns, there exists at most one row that intersects the three columns at a black square. Prove that

$$\frac{2k}{n} \leq \sqrt{8n-7}+1.$$

6. Let A, B, C and D be four points on a circle in this order. Assume that there is a point K on the segment AB such that BD bisects KC and AC bisects KD. Determine the minimal value that $\left|\frac{AB}{CD}\right|$ can take.

Good Luck!

IMO - Selection 2018

Third exam - 26 May 2018

Duration: 4.5 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

7. Let n be a positive integer. A sequence of $3n$ letters is called *Romanian* if the letters I, M and O appear exactly n times each. Define a *swap* to be the transposition of two adjacent letters. Prove that for any Romanian sequence X, there exists a Romanian sequence Y such that Y cannot be obtained from X using fewer than $\frac{3n^2}{2}$ swaps.

8. Determine all the integers $n \geq 2$ such that for every integer $0 \leq i, j \leq n$:

$$i + j \equiv \binom{n}{i} + \binom{n}{j} \pmod 2.$$

9. Let a, b, c, d be real numbers. Prove that

$$(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1) \geq \frac{9}{16}(a - b)(b - c)(c - d)(d - a).$$

Good Luck!

IMO - Selection 2018

Fourth exam - 27 May 2018

Duration: 4.5 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

10. Let ABC be a triangle, M the midpoint of BC and D a point on the line AB such that B lies between A and D. Let E be a point such that E and B are on different sides with respect to the line CD and such that $\angle EDC = \angle ACB$ and $\angle DCE = \angle BAC$. Let F be the intersection point of CE with the parallel line to DE through A. Let Z be the intersection point of AE and DF. Prove that the lines AC, BF and MZ intersect in a point.

11. Determine all the pairs (f, g) of functions $f, g\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

- $f(x) \geq 0$,
- $f(x + g(y)) = f(x) + f(y) + 2yg(x) - f(y - g(y))$.

12. David and Linus play the following game: David chooses a subset Q of $\{1, \ldots, 2018\}$. Then Linus chooses a natural number a_1 and computes inductively the numbers $a_2, \ldots, a_{2018}$ with a_{n+1} being the product of all positive divisors of a_n.

Let P be the set of integers $k \in \{1, \ldots, 2018\}$ for which a_k is a perfect square. Linus wins if $P = Q$. Otherwise David wins. Determine which player has a winning strategy.

Good Luck!