MATHEMATICAL.
OLYMPIAD.CH
MATHEMATIK-OLYMPIADE
OLYMPIADES DE MATHÉMATIQUES
OLIMPIADI DELLA MATEMATICA

Final round 2019

First exam

1 March 2019

Duration: 4 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

1. Let A be a point and k a circle through A. Let B and C be two other points on k. Let now X be the intersection of the angle bisector of $\angle ABC$ with k. Further, let Y be the reflection of A in point X, and D be the intersection of the line YC and k. Show that the point D does not depend on the choice of the points B and C on the circle k.

2. Let $\mathbb{P}$ be the set of all primes and let M be a subset of $\mathbb{P}$, having at least three elements, and such that the following property is satisfied: For any positive integer k and for any subset $A = \{p_1, p_2, \ldots, p_k\}$ of M with $A \neq M$, all of the prime factors of the number $p_1 \cdot p_2 \cdot \ldots \cdot p_k - 1$ are contained in M. Prove that $M = \mathbb{P}$.

3. Determine all periodic sequences $x_1, x_2, x_3, \ldots$ of positive real numbers such that for all positive integers n

$$x_{n+2} = \frac{1}{2}\left(\frac{1}{x_{n+1}} + x_n\right).$$

4. Let n be a positive integer. We place $n+1$ bowls in a row and number them from left to right by the numbers $0, 1, \ldots, n$. In the beginning, n stones lie in bowl 0 and no stone lies in any of the other bowls. Sisyphus wants to move all n stones to bowl n. To do this, in one step Sisyphus moves exactly one stone from a bowl containing $k \geq 1$ stones at most k bowls to the right. Let T be the minimal number of steps which Sisyphus needs to move all stones to bowl n. Show that:

$$T \geq \left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \ldots + \left\lceil \frac{n}{n} \right\rceil.$$

Remark: As usual $\lceil x \rceil$ stands for the least integer larger than or equal to x.

Good Luck!

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2 March 2019

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5. A group of children sits in a circle. At the beginning, each child has an even number of sweets. In each step, each child has to give half of its sweets to the child sitting on its right. If after a step a child has an odd number of sweets, it gets one additional sweet from the teacher. Show that after a finite number of steps all children have the same amount of sweets.

6. Show that there exists no function $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $m, n \in \mathbb{Z}$

$$f(m + f(n)) = f(m) - n.$$

7. Let ABC be a triangle with $\angle CAB = 2 \cdot \angle ABC$. Assume that there exists a point D in the interior of triangle ABC such that $AD = BD$ and $CD = AC$. Show that $\angle ACB = 3 \cdot \angle DCB$.

8. A positive integer $n \geq 2$ is called *resistant* if it is coprime to the sum of all its divisors (including 1 and n). What is the maximal number of consecutive resistant numbers?

Good Luck!