Preliminary round 2019

Lausanne, Lugano, Zurich
8 December 2018

Duration: 3 hours
Difficulty: The problems within one subject are orderd by difficulty.
Points: Each problem is worth 7 points.

Geometry

G1) Let k be a circle with center O and let A, B and C be three points on k with $\angle ABC > 90°$. The angle bisector of $\angle AOB$ intersects the circumcircle of triangle BOC again at D. Prove that D lies on the line AC.

G2) Let k_1 be a circle and l a line that intersects k_1 in two distinct points A and B. Let k_2 be another circle outside of k_1 that touches k_1 at C and l at D. Let T be the second intersection of k_1 and the line CD. Prove that $AT = TB$.

Combinatorics

C1) For a positive integer n, Timothy writes all the $2^n - 1$ nonempty subsets of $\{1, 2, \ldots, n\}$ on a line. Then, under each subset, he writes down the product of its elements. Lastly, he writes down the inverses of all numbers in the second line and computes their sum. What is the value of the sum (depending on n) obtained by Timothy?

Example: For $n = 3$, Timothy obtains:

$$\begin{array}{ccccccccccccccc} \{1\} & & \{2\} & & \{3\} & & \{1,2\} & & \{1,3\} & & \{2,3\} & & \{1,2,3\} & & \\ 1 & & 2 & & 3 & & 1\cdot 2 = 2 & & 1\cdot 3 = 3 & & 2\cdot 3 = 6 & & 1\cdot 2\cdot 3 = 6 & & \\ \frac{1}{1} & + & \frac{1}{2} & + & \frac{1}{3} & + & \frac{1}{2} & + & \frac{1}{3} & + & \frac{1}{6} & + & \frac{1}{6} & & = 3. \end{array}$$

C2) Let n be a positive integer. A volleyball team consisting of n men and n women is preparing to play. Every player is assigned one of the positions $1, 2, \ldots, 2n$ of which exactly the positions 1 and $n+1$ lie outside the court. During the game, the players rotate in such a way that the player on position i switches to position $i+1$ (respectively from position $2n$ to 1). In how many ways can the positions initially be assigned so that there are always at least $n-1$ women on the court, no matter how many rotations have occured?

Remark: Two initial positions are different, if at least one player is assigned to two different positions.

Number Theory

N1) Determine all pairs of positive integers (a, b) such that

$$ab + 2 = a^3 + 2b.$$

N2) Determine all positive integers $n \geq 2$ that can be written as

$$n = k^2 + d^2,$$

where k is the smallest divisor of n greater than 1 and d is any divisor of n.

Good luck!