MATHEMATICAL. OLYMPIAD.CH
MATHEMATIK-OLYMPIADE
OLYMPIADES DE MATHÉMATIQUES
OLIMPIADI DELLA MATEMATICA

Selection 2019

First exam

4 May 2019

Duration: 4.5 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

1. Let ABC be a triangle and D, E, F be the feet of the altitudes from A, B, C, respectively. Let H be the orthocenter of triangle ABC. The segments EF and AD intersect at G. Let K be the point on the circumcircle of ABC such that AK is a diameter of this circle. The line AK intersects BC at M. Prove that the lines GM and HK are parallel.

2. Determine the largest prime p such that there exist two positive integers a and b with

$$p = \frac{b}{2}\sqrt{\frac{a-b}{a+b}}.$$

3. Let S be a nonempty set of positive integers. Prove that at least one of the following two assertions holds:

(i) There exist distinct finite nonempty subsets F and G of S such that

$$\sum_{x\in F}\frac{1}{x} = \sum_{x\in G}\frac{1}{x}.$$

(ii) There exists a positive rational number $r < 1$ such that, for all finite nonempty subsets F of S,

$$\sum_{x\in F}\frac{1}{x} \neq r.$$

Good Luck!

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Selection 2019

Second exam
5 May 2019

Duration: 4.5 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

4. Let p be a prime. Determine all polynomials P with integer coefficients satisfying the following two conditions:

(i) $P(x) > x$ for all positive integers x.

(ii) If the sequence $(p_n)_{n\geq 0}$ is defined by $p_0 := p$ and $p_{n+1} := P(p_n)$ for every $n \geq 0$, then for every positive integer m, there exists $l \geq 0$ such that m divides p_l.

5. Let ABC be a triangle with $AB = AC$ and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the line PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

6. Let (a, b) be a pair of positive integers. Henning and Paul are playing a game: Initially, there are two piles of a and b stones, respectively, on a table. The pair (a, b) is called the *initial configuration* of the game. The players proceed as follows:

- The players alternate and Henning begins.
- In each turn, a player either removes a positive number of stones from one of the two piles or the same positive number of stones from both piles.
- The player who removes the last stone from the table wins the game.

Let A be the set of all positive integers a for which there exists a positive integer $b < a$ such that Paul has a winning strategy for the initial configuration (a, b). Order the elements of A increasingly as $a_1 < a_2 < \ldots$.

(a) Prove that the set A is infinite.

(b) Prove that the sequence $(m_k)_{k\geq 1}$ defined by $m_k := a_{k+1} - a_k$ for all $k \geq 1$ is not eventually periodic.

Remark: A sequence $(x_k)_{k\geq 1}$ is *eventually periodic* if there exists an integer $k_0 \geq 0$ such that the sequence $(x_{k+k_0})_{k\geq 1}$ is periodic.

Good Luck!

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Third exam

18 May 2019

Duration: 4.5 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

7. Prove that for all positive integers n, there exist two integers a and b such that

$$n \mid 4a^2 + 9b^2 - 1.$$

8. Let k, n and r be positive integers with $r < n$. Mickey has $kn + r$ black socks and $kn + r$ white socks. He wants to hang them on a straight clothesline in such a way that there are no $2n$ consecutive socks of which n of them are black and n of them are white. Show that he can do this if and only if $r \geq k$ and $n > k + r$ holds.

9. Let ABC be an acute triangle with $AB < AC$. Let E and F be the feet of the altitudes from B and C, respectively, and M the midpoint of the segment BC. The tangent line to the circumcircle of ABC at A intersects the line BC at P. The line parallel to BC through A intersects the line EF at Q. Prove that the line PQ is perpendicular to the line AM.

Good Luck!

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Selection 2019

Fourth exam
19 May 2019

Duration: 4.5 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

10. Let $n \geq 5$ be an integer. A store sells juggling balls of n different colours. Each of $n+1$ children buys three juggling balls of three different colours, but no two children buy exactly the same combination of colours. Show that there are at least two children who bought exactly one ball of the same colour.

11. Let n be a positive integer. Determine whether there exists a real number $\epsilon > 0$ (depending on n) such that, for all positive real numbers $x_1, x_2, \ldots, x_n$, we have

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq (1-\epsilon) \cdot \frac{x_1 + x_2 + \cdots + x_n}{n} + \epsilon \cdot \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

12. Define the sequence $(a_n)_{n \geq 0}$ of integers by $a_n := 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of two or more distinct terms of the sequence, as well as infinitely many terms which cannot be expressed in such a way.

Good Luck!