MATHEMATICAL.
OLYMPIAD.CH
MATHEMATIK-OLYMPIADE
OLYMPIADES DE MATHÉMATIQUES
OLIMPIADI DELLA MATEMATICA

Final round 2020

First exam

28 February 2020

Duration: 4 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

1. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$

$$f(m) + f(n) \mid m + n.$$

2. Let ABC be an acute triangle. Denote by M_A, M_B and M_C the midpoints of sides BC, CA and AB, respectively. Let M'_A, M'_B and M'_C be respectively the midpoints of the minor arcs BC, CA and AB on the circumcircle of ABC. Let P_A be the intersection of the lines M_BM_C and the perpendicular to $M'_BM'_C$ containing A. Let P_B and P_C be defined analogously. Prove that the lines M_AP_A, M_BP_B and M_CP_C meet at a point.

3. We are given n distinct rectangles in the plane. Prove that between the $4n$ interior right angles formed by these rectangles at least $4\sqrt{n}$ are distinct.

4. Let φ denote the Euler phi-function. Prove that for every positive integer n

$$2^{n(n+1)} \mid 32 \cdot \varphi\left(2^{2^n} - 1\right).$$

Good Luck!

Final round 2020

Second exam

29 February 2020

Duration: 4 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

5. Find all the positive integers a, b, c such that

$$a! \cdot b! = a! + b! + c!.$$

6. Let $n \geq 2$ be an integer. Consider the following game: Initially, k stones are distributed among the n^2 squares of an $n \times n$ chessboard. A move consists of choosing a square containing at least as many stones as the number of its adjacent squares (two squares are *adjacent* if they share a common edge) and moving one stone from this square to each of its adjacent squares.

Determine all positive integers k such that:

(a) There is an initial configuration with k stones such that no move is possible.

(b) There is an initial configuration with k stones such that an infinite sequence of moves is possible.

7. Let $ABCD$ be an isoceles trapezoid with $AD > BC$. Let X be the intersection point of the angle bisector of $\angle BAC$ and BC. Let E be the intersection point of DB with the parallel to the angle bisector of $\angle CBD$ through X and let F be the intersection point of DC and the parallel to the angle bisector of $\angle DCB$ through X. Prove that $AEFD$ is a cyclic quadrilateral.

8. Let n be a positive integer. Let $x_1 \leq x_2 \leq \ldots \leq x_n$ be a sequence of real numbers such that $x_1 + x_2 + \ldots + x_n = 0$ and $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Prove that $x_1 x_n \leq -1/n$.

Good Luck!