Final round 2020

Solutions

Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes.

Below you will find the elementary solutions known to correctors. Alternative solutions are presented in a complementary section at the end of each problem. Students are encouraged to use any methods at their disposal when training at home, but should be wary of attempting to find alternative solutions using methods they do not feel comfortable with under exam conditions as they risk losing valuable time.

1. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$

$$f(m) + f(n) \mid m + n.$$

Réponse: La seule solution est la fonction identité.

Solution 1(Arnaud): Soit f une solution. Avec $m = n = 1$, on obtient $f(1) \mid 1$ et donc $f(1) = 1$. Posons à présent $n = 1$. On obtient

$$f(m) + 1 \mid m + 1.$$

Il serait intéressant de rendre le côté droit premier. Posons donc $m = p - 1$ pour p un nombre premier. On a donc $f(p-1) + 1 \mid p$ et comme $f(p-1) + 1 > 1$, forcément $f(p-1) + 1 = p$ et donc $f(p-1) = p - 1$ pour tous les nombres premiers p.

Posons à présent $n = p - 1$, on a $f(m) + p - 1 \mid m + p - 1$ et donc

$$f(m) + p - 1 \mid m - f(m).$$

Dans cette dernière relation, on fixe m et on laisse p tendre vers l'infini. On obtient ainsi des diviseurs arbitrairement grands pour $m - f(m)$. Forcément $f(m) = m$.

On vérifie facilement que l'identité est une solution.

Solution 2(Ivan): Comme dans la première solution, on montre que $f(1) = 1$. On procède à présent par induction. Supposons que $f(m) = m$ pour tout $m \geq m_0$. On va montrer que $f(m_0 + 1) = m_0 + 1$.

Soit $m = m_0 + 1$ et $n = m_0$. On a

$$f(m_0 + 1) + m_0 \mid 2m_0 + 1.$$

On observe que $f(m_0 + 1) + m_0 \geq 1 + m_0 > (2m_0 + 1)/2$. Donc $f(m_0 + 1) + m_0$ est un diviseur de $2m_0 + 1$ et il est strictement plus grand que la moitié de $2m_0 + 1$ (qui est potentiellement le plus grand diviseur de $2m_0 + 1$ différent de $2m_0 + 1$). Forcément $f(m_0 + 1) + m_0 = 2m_0 + 1$ et ainsi $f(m_0 + 1) = m_0 + 1$.

Comme avant, on vérifie que l'identité est bien une solution.

Solution 3(Viviane): On procède également par induction. Avec $m = n$, on obtient $f(m) \mid m$ et donc $f(m) \leq m$. Supposons que $f(m) = m$ pour tout $m \leq m_0 - 1$. Supposons par l'absurde que $f(m_0) < m_0$, donc on peut poser $n = m_0 - f(m_0) \geq 1$ dans la relation de départ. Comme $m_0 - f(m_0) \leq m_0 - 1$, par hypothèse on a $f(m_0 - f(m_0)) = m_0 - f(m_0)$. Avec $m = m_0$ et $n = m_0 - f(m_0)$, on a

$$f(m_0) + m_0 - f(m_0) \mid 2m_0 - f(m_0)$$

et donc $m_0 \mid f(m_0)$. Contradiction. Ainsi $f(m_0) = m_0$ et on vérifie que l'identité est bien une solution.

Marking scheme

1P: $f(1) = 1$

(a) Solution 1

- 3P: $f(n) = n$ for infinitely many n
- 3P: conclude

(b) Solution 2 (or general inductive solutions)

- 3P: get a relation that allows you to conclude $f(m_0 + 1) = m_0 + 1$ (eg. $f(m_0 + 1) + m_0 \mid 2m_0 + 1$)
- 3P: conclude

(c) Solution 3

- 1P: $n \geq f(n)$
- 2P: plug in $n = m_0 - f(m_0)$ when $m_0 > f(m_0)$ in an inductive scheme
- 3P: conclude

-1P: not mentioning that one has to check that the identity is indeed a solution

2. Soit ABC un triangle aigu. Soient M_A, M_B et M_C les milieux respectifs des côtés BC, CA et AB. Soient M'_A, M'_B et M'_C les milieux respectifs des arcs mineurs BC, CA et AB sur le cercle circonscrit au triangle ABC. Soit P_A l'intersection de la droite M_BM_C et de la perpendiculaire à $M'_BM'_C$ par A. Les points P_B et P_C sont définis de manière analogue. Montrer que les droites M_AP_A, M_BP_B et M_CP_C se coupent en un point.

Solution 1 : (David) On montre d'abord que AP_A est la bissectrice de l'angle $\angle BCA$. Pour cela, on peut d'abord remarquer que M'_BM_B et M'_CM_C sont les médiatrices de AC et AB, respectivement, s'intersectant ainsi en O le centre du cercle circonscrit. Mais $\triangle OM'_BM'_C$ étant isocèle, on a alors

$$\angle M_CM'_CM'_B = \angle M_BM'_BM'_C \tag{1}$$

Introduisons maintenant $X = AP_A \cap M'_CM'_B$. On remarque alors que $\angle M'_CM_CA = 90° = M'_CXA$, et donc $AXM_cM'_C$ est un quadrilatère inscrit, en particulier $\angle M_CAX = \angle M_CM'_CX$, et symétriquement $\angle M_BAX = \angle M_BM'_BX$. On a alors

$$\angle BAP_A = \angle M_CAX = \angle M_CM'_CM'_B \overset{(1)}{=} \angle M_BM'_BM'_C = \angle M_BAX = \angle CAP_A$$

prouvant bien que AP_A est la bissectrice de $\angle BAC$. Par symétrie, on obtient que les points P_A, P_B et P_C se trouvent sur les bissectrices des angles $\angle BAC$, $\angle CBA$ et $\angle ACB$, respectivement. On veut maintenant conclure en utilisant Céva sur le triangle $\triangle M_AM_BM_C$. On peut alors invoquer un résultat sur la bissectrice, qu'on peut démontrer avec le lemme magique par exemple :

$$\frac{M_CP_A}{P_AM_B} = \frac{M_CA}{AM_B}$$

et on conclut alors par Céva sur le triangle $\triangle M_AM_BM_C$, puisqu'on a symétriquement

$$\frac{M_CP_A}{P_AM_B} \cdot \frac{M_BP_C}{P_CM_A} \cdot \frac{M_AP_B}{P_BM_C} = \frac{M_CA}{AM_B} \cdot \frac{M_BC}{CM_A} \cdot \frac{M_AB}{BM_C} = 1$$

Solution 2 : (Marco): Voici une autre façon de démontrer que AP_A est la bissectrice de $\angle BAC$. Pour cela, introduisons I le centre du cercle inscrit, et il nous suffira de montrer que $M'_BM'_C$ est la médiatrice de AI. Mais c'est une conséquence directe du lemme important suivant :

Lemme 1 *Soit $\triangle ABC$ un triangle quelconque et I le centre de son cercle inscrit. Soit de plus X le milieu de l'arc BC ne contenant pas A, sur le cercle circonscrit à $\triangle ABC$. Alors,*

$$XB = XI = XC$$

Proof. Par WUM, A, I et X sont alignés. On a alors

$$\angle XBI = \angle XBC + \angle CBI = \angle XAC + \angle ABI = \angle IAB + \angle BAI = \angle XIB$$

Le triangle $\triangle XBI$ est alors isocèle en X, prouvant le lemme. □

On obtient alors $M'_CA = M'_CI$ et $M'_BA = M'_BI$, montrant que $AI \perp M'_CM'_B$ et ainsi que AP_A est la bissectrice de $\angle BAC$. On conclut comme dans la solution 1.

Remarque Cet exercice est un cas particulier du théorème du "Cevian Nest" cf. [Euclidean Geometry in Mathematical Olympiads, Evan Chen, p. 57]

Marking scheme

- 1P: Remarquer que AP_A est la bissectrice de $\angle BAC$
- 2P: Prouver que AP_A est la bissectrice de $\angle BAC$
- 2P: Prouver un rapport entre des longueurs des triangles $\triangle M_AM_BM_C$ et $\triangle ABC$ comme $\frac{M_CP_A}{P_AM_B} = \frac{M_CA}{AM_B}$ ou $\frac{M_CP_A}{P_AM_B} = \frac{BP'_A}{P'_AC}$ (avec $P'_A = AP_A \cap BC$)
- 1P: Reformuler la conclusion avec le théorème de Céva
- 1P: Conclure

3. We are given n distinct rectangles in the plane. Prove that between the $4n$ interior right angles formed by these rectangles at least $4\sqrt{n}$ are distinct.

Solution: (David) First of all, let's make the whole picture easier to handle: We can split the rectangles into groups such that in each group, all sides of rectangles are parallel or perpendicular to each other. We also choose these groups to be maximal, in particular: For any two rectangles of different groups, they don't have parallel sides. We observe that this way, no two right angles of different groups can be the same, so we may count the right angles in each group and sum it up in the end. Say there are k different groups and denote the number of rectangles in the i-th group by n_i. We can choose a coordinate system that has axes parallel to the rectangles in the i-th group and from now on only consider rectangles in this group.

We can now introduce the following variables:

- A_i: number of distinct right angles that form a top left corner
- B_i: number of distinct right angles that form a top right corner
- C_i: number of distinct right angles that form a bottom left corner
- D_i: number of distinct right angles that form a bottom right corner

Since every rectangle in the i-th group is uniquely determined by one top left and one bottom right corner and for any two distinct rectangles, the combination of top left and bottom right corner must be different, we get the estimation

$$A_i \cdot D_i \geq n_i$$

The same argument for top right corners and bottom left corners yields

$$B_i \cdot C_i \geq n_i$$

Now, by applying AM-GM to both expressions on the left hand sides, we get

$$\left(\frac{A_i + D_i}{2}\right)^2 \geq A_i \cdot D_i \geq n_i \;\Rightarrow\; A_i + D_i \geq 2\sqrt{n_i}$$

$$\left(\frac{B_i + C_i}{2}\right)^2 \geq C_i \cdot B_i \geq n_i \;\Rightarrow\; B_i + C_i \geq 2\sqrt{n_i}$$

$$\Rightarrow\; A_i + B_i + C_i + D_i \geq 4\sqrt{n_i}$$

In other words, the number of distinct right angles in the i-th group is greater or equal than $4\sqrt{n_i}$. Summing over all groups, we get the inequality:

$$\sum_{i=1}^{k}(A_i + B_i + C_i + D_i) \geq \sum_{i=1}^{k} 4\sqrt{n_i}$$

On the left hand side, we have the total number of distinct right angles. So it remains to show $\sum_{i=1}^{k} 4\sqrt{n_i} \geq 4\sqrt{n}$, which is obvious by

$$\left(\sum_{i=1}^{k}\sqrt{n_i}\right)^2 \geq \sum_{i=1}^{k}\sqrt{n_i}^2 = \sum_{i=1}^{k} n_i = n \qquad \square$$

Remark: It is also possible to save some algebraic effort in the end by arguing at the beginning that we can rotate each group without increasing the number of distinct right angles. However, one must be aware that a rectangle could land on top of another congruent rectangle that way, so the argument gets a bit trickier than that.

Marking scheme:

(a) Dealing with the configuration

- 1P: Splitting up rectangles into appropriate groups or stating that we might assume them to be axis-parallel.
- 1P: A rigorous argument why it's enough to consider this case or, equivalently, the estimation of the sum in the end.

(b) Case of parallel sides

- 1P: distinguishing the four different types of right angles and stating that $A_i + B_i + C_i + D_i$ is the number we want to bound.
- 1P: $A_i \cdot D_i \geq n_i$
- 1P: $B_i \cdot C_i \geq n_i$
- 1P: Using AM-GM on one inequality
- 1P: Using AM-GM on the second inequality and conclude

4. Let φ denote the Euler phi-function. Prove that for every positive integer n

$$2^{n(n+1)} \mid 32 \cdot \varphi\left(2^{2^n} - 1\right).$$

Solution: (Valentin) We induct on n. The cases $n = 1, 2, 3$ can easily be checked by hand:

- $n = 1$: $2^2 \mid 32 \cdot 2$.
- $n = 2$: $2^6 \mid 2^5 \cdot \varphi(15) = 2^5 \cdot 2 \cdot 2^3$
- $n = 3$: $2^{12} \mid 2^5 \cdot \varphi(255) = 2^5 \cdot 2 \cdot 2^2 \cdot 2^4 = 2^{12}$

For $n \geq 4$ assume we the statement is true for all $1 \leq k < n$ and note that

$$\varphi\left(2^{2^n} - 1\right) = \varphi\Big(\left(2^{2^{n-1}} - 1\right)\left(2^{2^{n-1}} + 1\right)\Big) = \varphi\left(2^{2^{n-1}} - 1\right) \cdot \varphi\left(2^{2^{n-1}} + 1\right)$$

since $\gcd(2^{2^{n-1}} - 1, 2^{2^{n-1}} + 1) = 1$. But from our inductive assumption we know that

$$2^{(n-1)n} \mid 32 \cdot \varphi\left(2^{2^{n-1}} - 1\right)$$

so all that is left to prove is that

$$2^{2n} \mid \varphi\left(2^{2^{n-1}} + 1\right).$$

Take now any prime p that divides $2^{2^{n-1}} + 1$ and let d be the order of 2 mod p. We know that

$$2^{2^{n-1}} \equiv -1 \bmod p, \text{ squaring gives } 2^{2^n} \equiv 1 \bmod p.$$

By the properties of the order we therefore have

$$d \mid 2^n \text{ but } d \nmid 2^{n-1}.$$

This implies $d = 2^n$ and since we also have $d \mid p - 1$ we get

$$2^n \mid p - 1 \text{ and therefore } p \equiv 1 \bmod 2^n.$$

If we are also able to prove that $2^{2^{n-1}} + 1$ contains at least two different prime factors we would be done. This is because if p, q are two different such primes we can write

$$2^{2^{n-1}} + 1 = p^x \cdot q^y \cdot N$$

with N a positive integer and $p, q \nmid N$. Then

$$\varphi\left(2^{2^{n-1}} + 1\right) = (p-1)(q-1) \cdot p^{x-1} q^{x-1} \varphi(N) \equiv 0 \bmod 2^{2n}.$$

Assume now that $2^{2^{n-1}} + 1$ is instead a prime power, say p^x. It follows that

$$(p-1)\left(p^{x-1} + p^{x-2} \ldots + p + 1\right) = p^x - 1 = 2^{2^{n-1}}$$

It follows that x is odd since squares are $\equiv 0$ or $1 \bmod 4$ and using the fact that $p \equiv 1 \bmod 2^n$ we find

$$p^{x-1} + p^{x-2} \ldots + p + 1 \equiv x \bmod 2^n$$

implying $x = 1$. But then $2^{2^{n-1}} + 1$ is a prime and

$$\varphi\left(2^{2^{n-1}} + 1\right) = 2^{2^{n-1}}$$

and since $n \geq 4$ we have $2^{n-1} \geq 2n$ and we are done in this case as well. $\square$

Remarks: The fact that $p \mid 2^{n-1}+1 \implies p \equiv 1 \bmod 2^n$ for $n \geq 2$ can also be shown using induction or just quoted as a lemma. The contradiction in the case where $2^{2^{n-1}}+1$ is a prime power also follows from Catalan.

Marking Scheme:

- 1P: Using induction, reduction or factorisation to reduce the problem to something like $2^{2n} \mid \varphi\left(2^{2^{n-1}}+1\right)$ for large enough n. This includes the proof that $\gcd\left(2^{2^i}+1, 2^{2^j}+1\right)=1$ for $i \neq j$.
- 1P: Considering the order of 2 mod any factor of $2^{2^{n-1}}+1$.
- 1P: Proving that $2^n \mid p-1$.
- 1P: Having the idea of distinguishing the case where $2^{2^{n-1}}+1$ is a prime power and the case where it is divisible by two different primes.
- 1P: Finishing the case where $2^{2^{n-1}}+1$ is divisible by two different primes.
- 1P: Proving that $2^{2^{n-1}}+1$ is not a a prime power for large enough n.
- 1P: Treating the case where $2^{2^{n-1}}+1$ is prime including the small edge cases and concluding the Proof.
- -1P: If the small edge cases for n were missed.
- -1P: If any edge cases of stated theorems were missed.

5. Find all the positive integers a, b, c such that

$$a! \cdot b! = a! + b! + c!.$$

Answer: The only solution to this equation is $(a, b, c) = (3, 3, 4)$.

Solution: (Valentin and Tanish) Without loss of generality we assume $a \leq b$. We now divide the entire expression by $b!$ and get

$$a! = \frac{a!}{b!} + \frac{c!}{b!} + 1.$$

If $a < b$ then also $c < b$ since the right hand side should be an integer. But then

$$\frac{a!}{b!} + \frac{c!}{b!} + 1 < 3$$

and we get $a = 1$ or 2, both leading to a contradiction. We conclude that $a = b$. The problem is therefore reduces to

$$a! = \frac{c!}{a!} + 2 > 2$$

which implies $c \geq a \geq 3$. If $c \geq a + 3$ then the the right side is not divisible by 3 while the left side is. Otherwise if $a \leq c \leq a + 2$ we either have $c = a$ which does not lead to a solution, $c = a + 1 \Longrightarrow a! = a + 3$ and leads to the solution $(3, 3, 4)$, or $c = a + 2$ which leads to

$$a! = (a + 1)(a + 2) + 2 = a^2 + 3a + 4.$$

Here, the left side is strictly bigger than the right side for $a \geq 5$ and smaller values do not lead to any solutions.

Solution: (Louis)

On regroupe tous les termes contenant a ou b au côté gauche de l'équation pour obtenir

$$a!b! - a! - b! = c!$$

On suppose sans perte de généralité que $a \leq b$ et donc le côté gauche est divisible par $a!$. Il s'ensuit que le côté droit est également divisible par $a!$, donc $c \geq a$. Après division par $a!$ on obtient l'équation

$$b! - 1 - (a + 1) \cdots b = (a + 1) \cdots c.$$

Si b et c sont tous les deux strictement plus grands que a on devrait alors avoir que $a + 1$ divise 1 (car $a + 1$ divise tous les autres termes), ce qui est impossible car a est positif. Ainsi, on a soit $a = b$, soit $a = c$.

- Si $a = c$, l'équation originale peut être réécrite $(a! - 1) \cdot b! = 2a!$. Ainsi $(a! - 1)$ divise $2a!$, mais $\gcd(a! - 1, 2a!) = 2$, donc la seule solution est $a = 2$, qui implique alors $b! = 4$, et cette équation n'admet aucune solution. Ainsi l'équation originelle n'admet aucune solution si $a = c$.
- Si $a = b$, on obtient $a! - 2 = (a + 1) \cdots c$ et on termine comme dans la solution précédente.

Marking scheme

- 1P: Assume *WLOG* that $a \geq b$ or $b \geq a$.
- 1P: Use any technique to find a decent lower bound on c in terms of a and b.
- 1P: Use any technique to find a decent upper bound of one of the variables in term of the others. Therefore reduce to finitely many cases in two variables.
- 1P: Use any technique to again find a decent upper bound on one of the variables in terms of the other.
- 1P: Find the only solution $(3, 3, 4)$.
- 1P: Treat at least one of the cases that do not lead to a solution.
- 1P: Conclude the proof.

6. Let n be a positive integer. Consider the following game: Initially, k stones are distributed among the n^2 squares of an $n \times n$ chessboard. A move consists of choosing a square containing at least as many stones as the number of its adjacent squares (two squares are *adjacent* if they share a common edge) and moving one stone from this square to each of its adjacent squares.

Determine all positive integers k such that:

(a) There is an initial configuration with k stones such that no move is possible.

(b) There is an initial configuration with k stones such that an infinite sequence of moves is possible.

Answer: The possible values of k are all the values in the range $[2n^2 - 2n, 3n^2 - 4n]$.

Solution (Tanish)

Clearly, the first requirement imposes an *upper bound* on k (if there is an initial configuration with k stones where no moves are possible, then taking away stones from this configuration will not suddenly make a move possible) and the second requirement imposes a *lower bound* on k (similarly, if infinitely many moves are possible for a given k then we can just add stones and ignore their presence altogether).

First, we determine the upper bound. Note that if we place on each square a number of stones that is one less than the number of squares adjacent to it (so 1 on every corner, 2 on every edge and 3 on "inner" squares) then no moves are possible, as no square has enough stones for a move to be made. However, should we take any more than this number, then a pigeonhole principle on the squares shows that, for any given configuration with this many stones, a move is always possible because at least one square will have at least as many stones as it does adjacent squares. Calculating the total number of stones, we get $3n^2 - 4n$.

Now, we propose a lower bound for the second condition of $2n^2 - 2n$. A configuration with this many stones that has an infinite number of possible moves does indeed exist; place on all the white squares a number of stones equivalent to the number of squares bordering them. Then firstly perform a move on all of the white squares. After this has been done, each of the black squares will have a number of stones equivalent to the number of squares bordering them. After doing a move on each of the black squares, we find ourselves back at our original state, and proving there is a loop between the same state is sufficient to finish, as one has merely to repeat this loop indefinitely to have the desired infinite sequence.

Now let us show that $2n^2 - 2n$ is indeed a bound. If an infinite sequence is possible, as there are only finitely many configurations available to us, you must be able to go from some configuration to itself in a finite number of moves. To do so, you must apply a move on every square at least once (it can be proven that you have to apply the same number of moves to every square, but this is not necessary to finish) as if there are squares that are untouched, at least one square not used for a move is adjacent to a square that has been used and the net movement of stones to the unused square has to be positive, which is a contradiction with the fact the net movement should be 0 everywhere to return to the same state. Now, each stone can be labelled with an edge (here, an edge is defined as the border between two squares). To do so, all we do is the first time an edge is crossed by a stone, we assign that stone to the edge permanently, and henceforth the stone in question is made to cross over that edge exclusively. This can be done as to displace the stone you must play a move on its new square, but this will always enable you to return to its original square, and so forth. Clearly, only one stone is associated to each edge, and since every edge is crossed at least once, we must assign a stone to all edges, so we need at least $2n^2 - 2n$ stones. Note that not all stones are necessarily assigned an edge by the end; this can be thought of as a injective function from edges to stones.

Marking scheme

- 1P: Stating the upper bound or construction of 0-move configuration for $3n^2 - 4n$
- 1P: Any correct justification as to why this is the upper bound.
- 1P: Stating the construction with infinite moves for $2n^2 - 2n$
- 1P: Stating this is the lower bound
- 3P: Any correct justification as to why this is the lower bound.
  - 1P: At least one move per square with justification
  - 1P: Idea of looking for an injective function from edges to stones
  - 1P: Finding a well-defined function and concluding

7. Sei $ABCD$ ein gleichschenkliges Trapez mit $AD > BC$. Sei X der Schnittpunkt der Winkelhalbierenden von $\angle BAC$ und BC. Sei E der Schnittpunkt von DB mit der Parallelen zu der Winkelhalbierndden von $\angle CBD$ durch X und sei F der Schnittpunkt von DC mit der Parallelen zu der Winkelhalbierndden von $\angle DCB$ durch X. Zeige, dass $AEFD$ ein Sehenviereck ist.

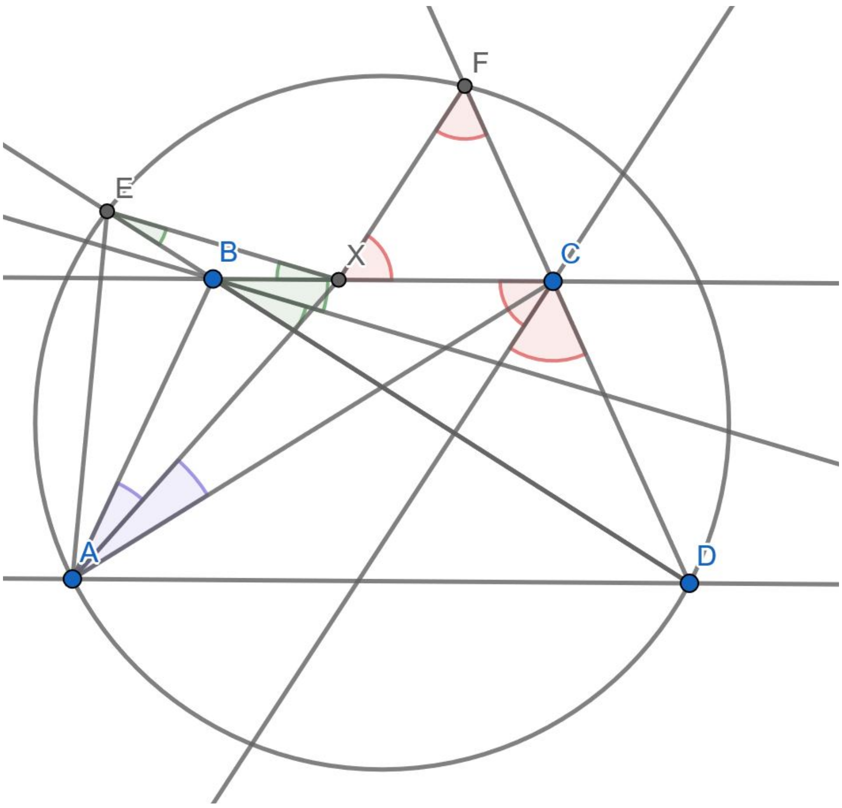
Lösung (Patrick)

Da $ABCD$ ein Sehnenviereck ist, hat man $\angle ABE = \angle ACF$.

Wenn man beweist, dass die Dreiecke $\triangle ABE$ und $\triangle ACF$ ähnlich sind, dann hat man, dass $\angle AED = \angle AEB = \angle AFC = \angle AFD$ also ist $AEFD$ ein Sehnenviereck. Man muss also nur noch zeigen, dass $\frac{AB}{BE} = \frac{AC}{CF}$.

Weil die Gerade EX parallel zur Winkelhalbieren ist, findet man $\angle XEB = \angle EXB$ und somit $BE = BX$ und auf genau gleiche Weise $CF = CX$.

Die Dreiecke AFC und AEB sind also genau dann ähnlich zueinander, wenn $\frac{AB}{BX} = \frac{AC}{CX}$. Diese Gleichung findet man aber mit dem magischen Lemma oder mit ähnlichen Dreiecken, da X der Schnittpunkt der Winkelhalbierenden von $\angle BAC$ mit BC ist.



Marking scheme

(a) 1P: Aufschreiben, dass $\angle BXE = \angle BEX$, $\angle CXF = \angle CFX$ oder $\angle ABE = ACF$.

(b) 1P: Zeige, dass $CF = CX$, $BE = BX$ und $\angle ABE = ACF$.

(c) 1P: Aufschreiben, dass $\frac{AB}{BX} = \frac{AC}{CX}$.

(d) 1P: Aufschreiben, dass $\frac{AB}{BE} = \frac{AC}{FC}$.

(e) 1P: Beweise, dass AFC ähnlich zu AEB.

(f) 2P: Feststellen, dass es ausreicht zu zeigen, dass AFC und AEB ähnlich zueinander sind.

8. Soit n un nombre entier strictement positif. Soient $x_1 \leq x_2 \leq \ldots \leq x_n$ des nombres réels tels que $x_1 + x_2 + \ldots + x_n = 0$ et $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Montrer que $x_1 x_n \leq -1/n$.

Solution 1 (Arnaud) On commence par élever au carré la condition que la somme des x_i est 0:

$$\underbrace{\sum x_i^2}_{=1} + \sum_{i \neq j} 2x_i x_j = \left(\sum x_i\right)^2 = 0$$

et donc on conclut que

$$\sum_{i \neq j} 2x_i x_j = -1.$$

Le but est de faire apparaître seulement des termes $x_1 x_n$. Dans ce but, on peut réécrire la somme ci-dessus comme

$$\begin{aligned} \sum_{i \neq j} 2x_i x_j &= 2x_1 x_n + \sum_{k=2}^{n-1} x_k x_1 + x_k x_n + x_k \underbrace{(x_1 + \ldots + x_{k-1} + x_{k+1} + \ldots + x_n)}_{=-x_k} \\ &= 2x_1 x_n + \sum_{k=2}^{n-1} x_1 x_k + x_n x_k - x_k^2. \end{aligned}$$

Il suffit donc de montrer que $x_1 x_k + x_n x_k - x_k^2 \geq x_1 x_n$ pour conclure. On propose deux arguments.

(a) (David's clever trick) L'inégalité se factorise en $(x_i - x_1)(x_n - x_1) \geq 0$ qui est clairement vraie.

(b) Observer que comme $x_1 + \ldots + x_n = 0$ et que toutes les variables ne peuvent pas être 0 simultanément (à cause de la seuxième condition), alors on doit avoir $x_1 < 0$ et $x_n > 0$.

On distingue deux cas:

i. Si $x_k \geq 0$, alors $x_n x_k - x_k^2 = x_k(x_n - x_k) \geq 0$. De plus, $x_1 x_k \geq x_1 x_n$ car $x_1 < 0$ et $x_n \geq x_k$. Donc on a bien $x_1 x_k + x_n x_k - x_k^2 \geq x_1 x_n$.

ii. Si $x_k \leq 0$, alors $x_1 x_k - x_k^2 = x_k(x_1 - x_k) \geq 0$. De plus, $x_n x_k \geq x_1 x_n$ car $x_n > 0$ et $x_k \geq x_1$. Donc dans ce cas aussi on conclut que $x_1 x_k + x_n x_k - x_k^2 \geq x_1 x_n$.

Solution 2 A nouveau, on a $x_1 < 0$ et $x_n > 0$. Dans cette solution, on trouve l'indice k tel que $x_i \leq 0$ pour $i \leq k$ et $x_i > 0$ pour $i > k$. On estime à présent la somme des carrés:

$$\sum_{i=1}^{k} x_i^2 \leq \sum_{i=1}^{k} x_1 x_i = -x_1 \sum_{i=k+1}^{n} x_i \leq (n-k)(-x_1)x_n,$$

et

$$\sum_{i=k+1}^{n} x_i^2 \leq x_n \sum_{i=k+1}^{n} x_i = x_n \sum_{i=1}^{k} -x_i \leq k(-x_1)x_n.$$

En sommant ces deux inégalités on obtient bien $1 \leq -n x_1 x_n$.

Marking scheme

(a) Solution 1

- 1P: get $2x_1x_n + 2\sum_{i\neq j,(i,j)\neq(1,n)} x_ix_j = -1$ (i.e. put the x_1x_n apart)
- 2P: get $2\sum_{i\neq j,(i,j)\neq(1,n)} x_ix_j = \sum_{i=2}^{n-1} x_1x_i + x_nx_i - x_i(\hat{x_i})$
- 1P: get $x_1x_i + x_nx_i - x_i(\hat{x_i}) = x_1x_i + x_nx_i - x_i^2$
- 2P: claim that $x_1x_i + x_nx_i - x_i^2 \geq x_ix_n$ is sufficient to conclude
- 2P: conclusion
  - 2P: factorisation
  - 1P each: distinguishing $x_i \geq 0$ and $x_i \neq 0$

(b) Solution 2

- 1P: locating zero, i.e. $x_1 \leq \ldots \leq x_k \leq 0 < x_{k+1} \leq \ldots \leq x_n$
- 1P: observing that $\sum_{i=1}^{k}(-x_i) = \sum_{i=k+1}^{n} x_i$
- 2P: bound one of the sums with the largest term, eg. $\sum_{i=1}^{k}(-x_i) \leq k(-x_1)$
- 2P: bound one of the sums of the squares with the largest term, eg. $\sum_{i=1}^{k} x_i^2 \leq x_ix_k$
- 1P: conclude