Preliminary round 2020

Lausanne, Lugano, Zürich
7 December 2019

Duration: 3 hours
Difficulty: The problems within each subject are ordered by difficulty.
Points: Each problem is worth 7 points.

Geometry

G1) Let k be a circle with center O. Let A, B, C and D be four distinct points on k in this order such that AB is a diameter of k. The circumcircle of the triangle COD intersects AC again in P. Show that OP and BD are parallel.

G2) Let ABC be a triangle with $AB > AC$. The angle bisectors at B and C meet at point I inside the triangle ABC. The circumcircle of the triangle BIC intersects AB again in X and AC again in Y. Show that CX is parallel to BY.

Combinatorics

C1) Consider a white 5×5 square composed of 25 unit squares. How many different ways are there to colour one or more unit squares black such that the resulting black area is a rectangle?

C2) The village of Roche has 2020 residents. One day, the famous mathematician Georges de Rham makes the following observations:

- Every villager knows someone else with the same age as them.
- For any group of 192 people in the village, there are always at least three of them that have the same age.

Prove that there must exist a group of 22 villagers that all have the same age.

Number Theory

N1) If $p \geq 5$ is a prime number, let q denote the smallest prime number such that $q > p$ and let n be the number of positive divisors of $p + q$ (1 and $p + q$ included).

a) Prove that no matter the choice of p, the number n is always at least 4.

b) Find the actual minimal value m that n can reach among all possible choices for p. That is:

- Give an example of a prime number p for which the value m is reached.
- Prove that there is no prime number p for which n is smaller than m.

N2) Let p be a prime number and a, b, c and n positive integers with $a, b, c < p$ such that the three assertions

$$p^2 \mid a + (n-1) \cdot b, \qquad p^2 \mid b + (n-1) \cdot c, \qquad p^2 \mid c + (n-1) \cdot a$$

hold. Show that n is not a prime number.

Good luck!